Part I

Introductory Session: SH 2–3, BM 2

Outline

- Elementary algebra
- Equations
- Summation notation

Real numbers

- $\mathbb{N}$ (natural numbers):
- $\mathbb{Z}$ (integers):
- $\mathbb{Q}$ (rational numbers): , where $a, b \in \mathbb{Z}, b \neq 0$
  - decimal system: $42.14 = \quad \cdot 10^1 + \quad \cdot 10^0 + \quad \cdot 10^{-1} + \quad \cdot 10^{-2}$
  - scientific notation: `2.34e-02`$= \quad \cdot 10^{\quad} = $
    Note: Here, `e` is not the Euler constant!
  - can be written as *finite* decimal fractions (see above) or *periodic* decimal fractions:
    $13/11 =$
- $\mathbb{R}$ (real numbers): *arbitrary* decimal fractions – all of the above, plus:
  $\sqrt{2}, \pi, 2^{\sqrt{2}}, 0.12112111211112\ldots$ and many (many) others

Integer Powers I

Definition:

$$a^n := \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$$

$$a^0 := \quad \text{for } a \neq 0$$

$$a^{-n} := \frac{\quad}{a^n} \text{ for } a \neq 0$$

Properties:

$$a^r \cdot a^s = a^{\quad}$$

$$(a^r)^s = a^{\quad}$$

Integer Powers II

Example

If $x^{-2}y^3 = 5$, compute $x^2y^{-3} + 2x^{-10}y^{15}$

Integer Powers III

Example

Suppose you deposit €1000 in a bank account paying 2% interest at the end of each year. How much do you have after 5 years?

Integer Powers IV

Example

Suppose you buy something for €$1000 \cdot 1.02^5$ which decreases in value (depreciates) by 2% per year. How much is it worth after 5 years?

Integer Powers V

Example

How much money should you have deposited in a bank 5 years ago at 2% yearly interest in order to buy something for €1000 today?

Some important rules of algebra I

$$
\begin{aligned}
-(a+b) &= -a \quad b \\
a(b+c) &= ab + \quad \\
(a+b)(c+d) &= ac + ad + bc + bd \\
(a+b)^2 &= \quad \\
(a-b)^2 &= \quad \\
(a+b)(a-b) &= \quad
\end{aligned}
$$

Some important rules of algebra II

Example

Expand and simplify: $(2t-1)(t^2-2t+1)$

Some important rules of algebra III

Example

Expand and simplify: $(a+1)^2+(a-1)^2-2(a+1)(a-1)$

Some important rules for fractions I

$$\frac{a \cdot \not{c}}{b \cdot \not{c}} = \frac{a}{b} \qquad \text{if } b \neq 0 \text{ and } c \neq 0$$

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{}{c}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{}{bd}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Some important rules for fractions II

Example

Simplify:

$$\frac{1}{1+x^{p-q}}+\frac{1}{1+x^{q-p}}$$

Fractional powers I

Find x such that $x^2 = a$?

$$a^{1/2} \cdot a^{1/2} = a^{1/2+1/2} = a^1 = a$$

Find x such that $x^n = a$?

$$a^{1/n} \cdot \ldots \cdot a^{1/n} = a^{1/n+\ldots+1/n} = a^1 = a$$

Notation:

$$a^{1/2} = \sqrt{a}$$

$$a^{1/n} = \sqrt[n]{a}$$

Fractional powers II

Example

Compute $\sqrt[3]{27}$, $(1/32)^{1/5}$, and $0.0001^{0.25}$.

Fractional powers III

Example

An amount of €5000 in an account has increased to €6000 in 20 years. What (constant) yearly interest rate p has been used?

Inequalities I

Example

Find what values of x satisfy $3x - 5 > x - 3$.

Inequalities II

Example

Find all x such that $|3x - 2| \leq 5$.

Outline

- Elementary algebra
- Equations
- Summation notation

Equations I

Equations are called *equivalent* if they have the same solutions, and

- adding/subtracting the same number
- multiplying (or dividing) by the same number $\neq 0$

to both sides of the equality sign constitutes an equivalence transformation. We often call doing fancy equivalence transformations *solving*.

Equations II

Example

Solve for x:

$$\sqrt{1+x} + \frac{ax}{\sqrt{1+x}} = 0$$

Equations III

Example

A firm manufactures a commodity that costs €20 per unit to produce. In addition, the firm has fixed costs of €2000. Each unit is sold for €75. How many units must be sold if the firm is to meet a profit target of €14 500?

Quadratic equations I

Find x, such that

$$ax^2 + bx + c = 0, \qquad \text{where } a, b, c \in \mathbb{R}$$

- Easy case 1: $a = 0$:
- Easy case 2: $b = 0$:
- Easy case 3: $c = 0$:

Quadratic equations II

General case: If $b^2 - 4ac \geq 0$ and $a \neq 0$, then

$$ax^2 + bx + c = 0 \quad \text{if and only if} \quad x = \frac{-b \pm \sqrt{\qquad}}{2a}$$

Quadratic equations III

Example

A producer faces the following demand: $P = 100 - 2Q$, where P stands for the price of a certain product and Q for the quantity of products sold. For what price is the total revenue $TR = P \cdot Q$ equal to zero?

Outline

- Elementary algebra
- Equations
- Summation notation

Summation notation I

$$\sum_{i=1}^{n} N_i := N_1 + N_2 + \ldots + N_n$$

Some important properties:

$$\sum_{i=1}^{n} (a_i + b_i) =$$

$$\sum_{i=1}^{n} c \cdot a_i =$$

Summation notation II

Example

Evaluate

$$\sum_{i=-2}^{3} (i+3)^i$$

Summation notation III

Example

Express in summation notation:

$$1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots - \frac{x^{79}}{80} + \frac{x^{80}}{81}$$

Part II

Functions of One Variable: SH 4, BM 2

Outline

- Functions (SH 4.1–4.3)
- Linear Functions (SH 4.4–4.5)
- Polynomials (SH 4.6–4.7)
- Power Functions (SH 4.8)
- Exponential and Logarithmic Functions (SH 4.9–4.10)

Functions I

A function is an assignment. The definition of a function f requires three objects to be specified:

1. a *domain* A,
2. a target set (*codomain*) B,
3. rule that assigns to any element of the domain *one* element of the codomain.

Notation: $f: A \to B$, $x \mapsto f(x)$

The range of a function $f: A \to B$ is the set $f(A) = \{f(x) | \, x \in A\}$.

Functions II

Example

Assign to each person in this room his/her age (in years).

Functions III

Example

Assign to each age (in years) the corresponding person in this room.

Functions IV

Example

Assign to each number its square.

Functions V

Example

Assign to each area of a square its side length.

Functions VI

Example

The total dollar cost of producing x units of a product is given by $C(x) = 100x\sqrt{x} + 500$.
Domain? Codomain? Range? Graph?

Functions VII

Example

The *absolute value* function is defined as follows:

$$\begin{array}{rcl} \mathbb{R} & \to & \mathbb{R}_0^+ \\ x & \mapsto & |x| \end{array}$$

with

$$|x| = \begin{cases} & \text{if} \quad x \geq 0 \\ & \text{if} \quad x < 0 \end{cases}$$

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Linear (affine) functions I

$$f : \begin{cases} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & ax + b \end{cases}$$

- $a \in \mathbb{R} \ldots$
- $b \in \mathbb{R} \ldots$

Linear (affine) functions II

Example

Suppose demand D for a good is a linear function of its price per unit P. When price is €10, demand is 300 units, and when price is €15, demand is 250 units. Find the demand function.

Linear (affine) functions III

Example

Suppose supply S for a good is a linear function of its price per unit P. When price is €10, supply is 100 units, and when price is €20, supply is 200 units. Find the supply function.

Linear (affine) functions IV

Example

Graph D and S and find the equilibrium price $P : D(P) = S(P)$.

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Polynomials I

Polynomial of degree n:

$$f : \begin{cases} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \end{cases}$$

- $a_n \in \mathbb{R} \backslash \{0\}$
- $a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$

Well “behaved” and well understood from a mathematical point of view. However, finding roots, maxima, etc. can be tedious to do *by hand*.

Knowing the roots is powerful.

Polynomials II

Example

$$f : \begin{cases} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & ax^2 + bx + c \end{cases}$$

- $a \in \mathbb{R} \setminus \{0\}$
- $b, c \in \mathbb{R}$

Its graph is a parabola that opens

- upwards if
- downwards if

Polynomials III

Example

Note that

$$ax^2 + bx + c = a\left(x + \quad\right)^2 - \frac{b^2 - 4ac}{4a}$$

Polynomials IV

Example

Let $f(x) = -\frac{1}{2}x^2 - x + \frac{3}{2}$.

- Graph?
- Minimum/Maximum?
- $x : f(x) = 0$?
- Show that $f(x) = -\frac{1}{2}(x \quad)(x \quad)$ and use this to study how the sign of $f(x)$ varies.

Polynomials V

Example

Factorize $f(x) = -\frac{1}{2}x^3 - x^2 + \frac{3}{2}x$.

Outline

- Functions (SH 4.1–4.3)
- Linear Functions (SH 4.4–4.5)
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Power functions I

$$f : \begin{cases} \mathbb{R}^+ \setminus \{0\} & \to & \mathbb{R} \\ x & \mapsto & Ax^r \end{cases}$$

- $A, r \in \mathbb{R}$

If $r > 0$, then we may allow the value 0 in the domain of f with $f(0) = 0$.

Power functions II

Example

Assume that the relationship between the size of houses s (in m^2) and their selling price P (in €) follows approximately

$$P(s) = 40\,000 \cdot s^{0.4}$$

Outline

- Functions (SH 4.1–4.3)
- Linear Functions (SH 4.4–4.5)
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Exponential functions I

$$f : \begin{cases} \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & Aa^t \end{cases}$$

- $A \in \mathbb{R}$, $a \in \mathbb{R}^+ \backslash \{0\}$
- Special case: The *natural exponential function* $t \mapsto \exp(t)$ where $A = 1$ and $a = e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718281828459045\ldots$

Exponential functions II

Example

The normal (or Gaussian) density function is given by

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Exponential functions III

Some properties of exponential functions:

For any $a > 0$, $x \in \mathbb{R}$ we have

E1 $a^x \cdot a^y = a^{x+y}$

E2 $a^{xy} = (a^x)^y = (a^y)^x$

E3 $a^{-x} = \left(\frac{1}{a}\right)^x$

E4 $a^0 = 1$

Exponential functions IV

General Interpretation:

- If $a = 1 + \frac{p}{100}$, where $p > 0$, and $A > 0$, then $f(t)$ will increase by $p\%$ for each unit increase in t.
- If $a = 1 - \frac{p}{100}$, where $0 < p < 100$, and $A > 0$, then $f(t)$ will decrease by $p\%$ for each unit increase in t.

Exponential functions V

Example

Assume that you invest €10 at an annual interest rate of 1%. Determine $f(t)$, the amount you have t years from now. How long does it take (approximately) for your investment to double, how long to quadruple?

The logarithm I

The doubling time of an exponential function $f(t) = Aa^t$ was defined as the time it takes for $f(t)$ to become twice as large. In order to find the doubling time t^*, we must solve the equation $a^{t^*} = 2$ for t^*:

Logarithm function

For any positive number x,

$$a^{\log_a x} = x$$

Thus, $\log_a x$ is the power of a you need to get x:

$$f : \begin{cases} \mathbb{R}^+ \backslash \{0\} & \to & \mathbb{R} \\ x & \mapsto & \log_a x \end{cases}$$

Note: Sometimes, we write $\log$ (or $\ln$). This is the notation for $\log_e x$.

The logarithm II

Example

Find $\ln 1$, $\ln e$, $\ln 1/e$, $\ln 4$, and $\ln(-6)$.

The logarithm III

Some important rules for logarithms for $x, y > 0$ and $p \in \mathbb{R}$:

L1 $\log xy =$

L2 $\log x^p =$

L3 $\ln e^x =$

L4 $e^{\ln x} =$

What is the rule for $\ln(x) - \ln(y)$?

The logarithm IV

Example

Simplify $\exp(\ln x^2 - 2\ln y)$.

The logarithm V

Example

How long does it take for an amount x to double at a yearly interest rate of $i \in \{1, 2, 3\}$ per cent? Verify the “rule of 70“!

The logarithm VI

Example

Find the mistake in the following "proof", showing that 2 is smaller than 1.

$$1/4 < 1/2 \Leftrightarrow$$
$$\ln(1/4) < \ln(1/2) \Leftrightarrow$$
$$\ln((1/2)^2) < \ln(1/2) \Leftrightarrow$$
$$2\ln(1/2) < \ln(1/2) \Leftrightarrow$$
$$2 < 1$$

Part III

More About Functions: SH 5

Outline

- Transformations of Graphs
- New Functions from Old
- Inverse Functions
- Common Properties of Functions

Transformations of graphs I

- Replacing $f(x)$ with $f(x) + c$ moves the graph of f c units
- Replacing $f(x)$ with $f(x + c)$ moves the graph of f c units
- Replacing $f(x)$ with $cf(x)$ corresponds to a stretch ($c < 0$ gives an x-axis flip)
- Replacing $f(x)$ with $f(cx)$ corresponds to a compression ($c < 0$ gives a y-axis flip)

Transformations of graphs II

Example

Suppose a person earning y euros in a given year pays

$$T(y) = \begin{cases} \max\left(0, \frac{y^2}{1{,}000\,000} - 100\right) & \text{if } 0 \leq y < 100\,000 \\ 9\,900 + (y - 100\,000)/4 & \text{if } y \geq 100\,000 \end{cases}$$

euros that year in income tax. Illustrate graphically!

Transformations of graphs III

Transformations of graphs IV

Example

To reduce taxes,

- A suggests to allow every individual to deduct $10\,000$ euros before the tax is calculated.
- B suggests to allow every individual to deduct 5% before the tax is calculated.
- C suggests to calculate the income tax on the full amount and then to allow each person a "tax credit" of 1000 euros.
- D suggests to calculate the income tax on the full amount and then to allow each person a "tax credit" of 10%.

Visualize and comment.

Outline

- Transformations of Graphs
- New Functions from Old
- Inverse Functions
- Common Properties of Functions

New functions from old I

Sums, differences, products, and quotients of functions can be easily defined.

- The sum of f and g, $f+g$, is given by $(f+g)(x) = f(x) + g(x)$.
- The difference of f and g, $f-g$, is given by $(f-g)(x) = f(x) - g(x)$.
- The product of f and g, $f \cdot g$ or fg, is given by $(f \cdot g)(x) = f(x)g(x)$.
- The quotient of f and g, f/g, is given by $(f/g)(x) = f(x)/g(x)$.

New functions from old II

If $g : A \to B$ and $f : B \to C$, then the *composition* is defined by

$$f \circ g : \begin{cases} A & \to & C \\ x & \mapsto & f(g(x)) \end{cases}$$

The function g is often called the *kernel* or *interior/inner function*, while f is called the *exterior/outer function*.

Note: In general,

- $f \circ g \neq f \cdot g$ (and thus $f^2 \neq f \circ f$)
- $f \circ g \neq g \circ f$

New functions from old III

Example

If $f(x) = 3x - x^3$ and $g(x) = x^3$, compute and visualize $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, $(f/g)(x)$, $f(g(x))$, $g(f(x))$ and evaluate $f(g(1))$, and $g(f(1))$.

Outline

- Transformations of Graphs
- New Functions from Old
- Inverse Functions
- Common Properties of Functions

Injections, Surjections, Bijections I

- A function f is called *injective* if

- A function is called *surjective* if its range is equal to its codomain
- A function is called *bijective* if it is and

Injections, Surjections, Bijections II

Example

The function $f : \mathbb{R} \to \mathbb{R}; x \mapsto x^2$ is neither injective nor surjective.

Inverse functions I

Let $f : A \to B$ be a bijective function. A function $g : B \to A$ is called the *inverse* of f if

$$g \circ f = Id_A \qquad \text{(i.e. } g(f(x)) = x\text{)}$$

Notes:

- If $g \circ f = Id_A$ then $f \circ g = Id_B$
- The most common notation for the inverse of f is f^{-1}
- Don't confuse f^{-1} (the inverse) with $1/f$ (the reciprocal)!

Inverse functions II

Example

Find the inverse of $f : [-2, 2] \to [-9, 7]; x \mapsto x^3 - 1$. Plot f and f^{-1}.

Inverse functions III

Example

Consider f defined by $f(x) = 4\ln(\sqrt{x+4} - 2)$. Determine natural domain, range, and inverse.

Outline

- Transformations of Graphs
- New Functions from Old
- Inverse Functions
- Common Properties of Functions

Symmetry

A function f is called

- *symmetric about* a if for all x it holds that

$$f(a+x) = f(\qquad)$$

- *even* if f is symmetric about 0, i.e.

- *odd* if for all x it holds that

$$f(-x) = \qquad$$

Monotonicity

A function f is called

- *increasing* if $a < b \Rightarrow f(a) \quad f(b)$
- *decreasing* if $a < b \Rightarrow f(a) \quad f(b)$

for all values a, b on a suitable interval.

The function is called *strictly* increasing or decreasing (on that interval) if the corresponding strict inequalities hold.

A function is said to be *(strictly) monotonic* if it is either (strictly) increasing or (strictly) decreasing (on that interval).

A *local extremum* is a point where a function changes .

Convexity I

A function f is called

- *conVex* if
$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b)$$
- *concAve* if
$$f(\lambda a + (1-\lambda)b) \geq \lambda f(a) + (1-\lambda) f(b)$$

for all values a, b on a suitable interval and $\lambda \in [0,1]$.

An *inflection point* is a point where a graph changes .

Convexity II

Example

Discuss the convexity of f with $f(x) = x^2$.

Let $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$. We have:

$$\begin{aligned}
&(\lambda x_1 + (1-\lambda)x_2)^2 - \lambda x_1^2 - (1-\lambda)x_2^2 \\
&= (\lambda^2 - \lambda)x_1^2 + 2(\lambda - \lambda^2)x_1 x_2 + (\lambda^2 - \lambda)x_2^2 \\
&= (\lambda^2 - \lambda)(x_1 - x_2)^2 \\
&\leq 0
\end{aligned}$$

Hence, $f(x) = x^2$ is a convex function.