

---

## Part I

Introductory Session: SH 2–3, BM 2

# Outline

---

- Elementary algebra
- Equations
- Summation notation

# Real numbers

- $\mathbb{N}$  (natural numbers): [redacted]
- $\mathbb{Z}$  (integers): [redacted]
- $\mathbb{Q}$  (rational numbers): [redacted], where  $a, b \in \mathbb{Z}, b \neq 0$ 
  - decimal system:  $42.14 = [redacted] \cdot 10^1 + [redacted] \cdot 10^0 + [redacted] \cdot 10^{-1} + [redacted] \cdot 10^{-2}$
  - scientific notation:  $2.34e-02 = [redacted] \cdot 10^{[redacted]} = [redacted]$   
Note: Here, e is not the Euler constant!
  - can be written as *finite* decimal fractions (see above) or *periodic* decimal fractions:  
 $13/11 = [redacted]$
- $\mathbb{R}$  (real numbers): *arbitrary* decimal fractions – all of the above, plus:  
 $\sqrt{2}, \pi, 2\sqrt{2}, 0.12112111211112\dots$  and many (many) others

# Integer Powers I

Definition:

$$a^n := \underbrace{a \cdot a \cdot a \cdot \dots \cdot a}_{n \text{ factors}}$$

$$a^0 := \square \text{ for } a \neq 0$$

$$a^{-n} := \frac{\square}{a^n} \text{ for } a \neq 0$$

Properties:

$$a^r \cdot a^s = a^{\square}$$

$$(a^r)^s = a^{\square}$$

## Integer Powers II

### Example

If  $x^{-2}y^3 = 5$ , compute  $x^2y^{-3} + 2x^{-10}y^{15}$

# Integer Powers III

## Example

Suppose you deposit €1000 in a bank account paying 2% interest at the end of each year. How much do you have after 5 years?

# Integer Powers IV

## Example

Suppose you buy something for €  $1000 \cdot 1.02^5$  which decreases in value (depreciates) by 2% per year. How much is it worth after 5 years?

# Integer Powers V

## Example

How much money should you have deposited in a bank 5 years ago at 2% yearly interest in order to buy something for €1000 today?

# Some important rules of algebra I

---

$$-(a + b) = -a \blacksquare b$$

$$a(b + c) = ab + \blacksquare$$

$$(a + b)(c + d) = ac + ad + bc + bd$$

$$(a + b)^2 = \blacksquare$$

$$(a - b)^2 = \blacksquare$$

$$(a + b)(a - b) = \blacksquare$$

## Some important rules of algebra II

### Example

Expand and simplify:  $(2t - 1)(t^2 - 2t + 1)$

## Some important rules of algebra III

### Example

Expand and simplify:  $(a + 1)^2 + (a - 1)^2 - 2(a + 1)(a - 1)$

## Some important rules for fractions I

$$\frac{a \cdot \cancel{c}}{b \cdot \cancel{c}} = \frac{a}{b} \quad \text{if } b \neq 0 \text{ and } c \neq 0$$

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{\text{[ ]}}{c}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{\text{[ ]}}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{\text{[ ]}}{bd}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

## Some important rules for fractions II

### Example

Simplify:

$$\frac{1}{1 + x^{p-q}} + \frac{1}{1 + x^{q-p}}$$

# Fractional powers I

---

Find  $x$  such that  $x^2 = a$ ?

$$a^{1/2} \cdot a^{1/2} = a^{1/2+1/2} = a^1 = a$$

Find  $x$  such that  $x^n = a$ ?

$$a^{1/n} \cdot \dots \cdot a^{1/n} = a^{1/n+\dots+1/n} = a^1 = a$$

Notation:

$$a^{1/2} = \sqrt{a}$$

$$a^{1/n} = \sqrt[n]{a}$$

## Fractional powers II

### Example

Compute  $\sqrt[3]{27}$ ,  $(1/32)^{1/5}$ , and  $0.0001^{0.25}$ .

## Fractional powers III

### Example

An amount of €5000 in an account has increased to €6000 in 20 years. What (constant) yearly interest rate  $p$  has been used?

# Inequalities I

## Example

Find what values of  $x$  satisfy  $3x - 5 > x - 3$ .

## Inequalities II

### Example

Find all  $x$  such that  $|3x - 2| \leq 5$ .

# Outline

---

- Elementary algebra
- Equations
- Summation notation

# Equations I

---

Equations are called *equivalent* if they have the same solutions, and

- adding/subtracting the same number
- multiplying (or dividing) by the same number  $\neq 0$

to both sides of the equality sign constitutes an equivalence transformation. We often call doing fancy equivalence transformations *solving*.

## Equations II

### Example

Solve for  $x$ :

$$\sqrt{1+x} + \frac{ax}{\sqrt{1+x}} = 0$$

## Equations III

### Example

A firm manufactures a commodity that costs €20 per unit to produce. In addition, the firm has fixed costs of €2000. Each unit is sold for €75. How many units must be sold if the firm is to meet a profit target of €14 500?

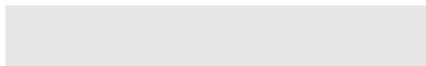
# Quadratic equations I

---

Find  $x$ , such that

$$ax^2 + bx + c = 0, \quad \text{where } a, b, c \in \mathbb{R}$$

■ Easy case 1:  $a = 0$ :



■ Easy case 2:  $b = 0$ :



■ Easy case 3:  $c = 0$ :



## Quadratic equations II

---

General case: If  $b^2 - 4ac \geq 0$  and  $a \neq 0$ , then

$$ax^2 + bx + c = 0 \quad \text{if and only if} \quad x = \frac{-b \pm \sqrt{\quad}}{2a}$$

## Quadratic equations III

### Example

A producer faces the following demand:  $P = 100 - 2Q$ , where  $P$  stands for the price of a certain product and  $Q$  for the quantity of products sold. For what price is the total revenue  $TR = P \cdot Q$  equal to zero?

# Outline

---

- Elementary algebra
- Equations
- Summation notation

# Summation notation I

---

$$\sum_{i=1}^n N_i := N_1 + N_2 + \dots + N_n$$

Some important properties:

$$\sum_{i=1}^n (a_i + b_i) = \text{[gray box]}$$

$$\sum_{i=1}^n c \cdot a_i = \text{[gray box]}$$

# Summation notation II

## Example

Evaluate

$$\sum_{i=-2}^3 (i+3)^i$$

## Summation notation III

### Example

Express in summation notation:

$$1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots - \frac{x^{79}}{80} + \frac{x^{80}}{81}$$

---

## Part II

# Functions of One Variable: SH 4, BM 2

# Outline

---

- Functions (SH 4.1–4.3)
- Linear Functions (SH 4.4–4.5)
- Polynomials (SH 4.6–4.7)
- Power Functions (SH 4.8)
- Exponential and Logarithmic Functions (SH 4.9–4.10)

# Functions I

---

A *function is an assignment*. The definition of a function  $f$  requires three objects to be specified:

1. a *domain*  $A$ ,
2. a target set (*codomain*)  $B$ ,
3. rule that assigns to any element of the domain *one* element of the codomain.

Notation:  $f : A \rightarrow B, x \mapsto f(x)$

The range of a function  $f : A \rightarrow B$  is the set  $f(A) = \{f(x) \mid x \in A\}$ .

# Functions II

---

## Example

Assign to each person in this room his/her age (in years).

# Functions III

---

## Example

Assign to each age (in years) the corresponding person in this room.

# Functions IV

---

## Example

Assign to each number its square.

# Functions V

---

## Example

Assign to each area of a square its side length.

# Functions VI

## Example

The total dollar cost of producing  $x$  units of a product is given by  $C(x) = 100x\sqrt{x} + 500$ .  
Domain? Codomain? Range? Graph?

# Functions VII

## Example

The *absolute value* function is defined as follows:

$$\begin{aligned}\mathbb{R} &\rightarrow \mathbb{R}_0^+ \\ x &\mapsto |x|\end{aligned}$$

with

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

# Outline

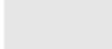
---

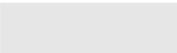
- Functions (SH 4.1–4.3)
- **Linear Functions (SH 4.4–4.5)**
- Polynomials (SH 4.6–4.7)
- Power Functions (SH 4.8)
- Exponential and Logarithmic Functions (SH 4.9–4.10)

# Linear (affine) functions I

---

$$f : \begin{cases} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & ax + b \end{cases}$$

■  $a \in \mathbb{R} \dots$  

■  $b \in \mathbb{R} \dots$  

## Linear (affine) functions II

### Example

Suppose demand  $D$  for a good is a linear function of its price per unit  $P$ . When price is €10, demand is 300 units, and when price is €15, demand is 250 units. Find the demand function.

## Linear (affine) functions III

### Example

Suppose supply  $S$  for a good is a linear function of its price per unit  $P$ . When price is €10, supply is 100 units, and when price is €20, supply is 200 units. Find the supply function.

## Linear (affine) functions IV

### Example

Graph  $D$  and  $S$  and find the equilibrium price  $P : D(P) = S(P)$ .

# Outline

---

- Functions (SH 4.1–4.3)
- Linear Functions (SH 4.4–4.5)
- **Polynomials (SH 4.6–4.7)**
- Power Functions (SH 4.8)
- Exponential and Logarithmic Functions (SH 4.9–4.10)

# Polynomials I

Polynomial of degree  $n$ :

$$f : \begin{cases} \mathbb{R} & \rightarrow \mathbb{R} \\ x & \mapsto a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \end{cases}$$

- $a_n \in \mathbb{R} \setminus \{0\}$
- $a_{n-1}, \dots, a_1, a_0 \in \mathbb{R}$

Well “behaved” and well understood from a mathematical point of view. However, finding roots, maxima, etc. can be tedious to do *by hand*.

*Knowing* the roots is powerful.

# Polynomials II

## Example

$$f : \begin{cases} \mathbb{R} & \rightarrow \mathbb{R} \\ x & \mapsto ax^2 + bx + c \end{cases}$$

- $a \in \mathbb{R} \setminus \{0\}$
- $b, c \in \mathbb{R}$

Its graph is a parabola that opens

- upwards if
- downwards if

# Polynomials III

## Example

Note that

$$ax^2 + bx + c = a \left( x + \square \right)^2 - \frac{b^2 - 4ac}{4a}$$

# Polynomials IV

## Example

Let  $f(x) = -\frac{1}{2}x^2 - x + \frac{3}{2}$ .

- Graph?
- Minimum/Maximum?
- $x : f(x) = 0$ ?
- Show that  $f(x) = -\frac{1}{2}(x - \text{ }) (x - \text{ })$  and use this to study how the sign of  $f(x)$  varies.

# Polynomials V

## Example

Factorize  $f(x) = -\frac{1}{2}x^3 - x^2 + \frac{3}{2}x$ .

# Outline

---

- Functions (SH 4.1–4.3)
- Linear Functions (SH 4.4–4.5)
- Polynomials (SH 4.6–4.7)
- **Power Functions (SH 4.8)**
- Exponential and Logarithmic Functions (SH 4.9–4.10)

# Power functions I

---

$$f : \begin{cases} \mathbb{R}^+ \setminus \{0\} & \rightarrow \mathbb{R} \\ x & \mapsto Ax^r \end{cases}$$

- $A, r \in \mathbb{R}$

If  $r > 0$ , then we may allow the value 0 in the domain of  $f$  with  $f(0) = 0$ .

## Power functions II

### Example

Assume that the relationship between the size of houses  $s$  (in  $m^2$ ) and their selling price  $P$  (in €) follows approximately

$$P(s) = 40\,000 \cdot s^{0.4}$$

# Outline

---

- Functions (SH 4.1–4.3)
- Linear Functions (SH 4.4–4.5)
- Polynomials (SH 4.6–4.7)
- Power Functions (SH 4.8)
- Exponential and Logarithmic Functions (SH 4.9–4.10)

# Exponential functions I

---

$$f : \begin{cases} \mathbb{R} & \rightarrow \mathbb{R} \\ t & \mapsto Aa^t \end{cases}$$

- $A \in \mathbb{R}$ ,  $a \in \mathbb{R}^+ \setminus \{0\}$
- Special case: The *natural exponential function*  $t \mapsto \exp(t)$  where  $A = 1$  and  $a = e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718281828459045 \dots$

## Exponential functions II

### Example

The normal (or Gaussian) density function is given by

$$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

## Exponential functions III

---

Some properties of exponential functions:

For any  $a > 0$ ,  $x \in \mathbb{R}$  we have

E1  $a^x \cdot a^y = a^{x+y}$

E2  $a^{xy} = (a^x)^y = (a^y)^x$

E3  $a^{-x} = \left(\frac{1}{a}\right)^x$

E4  $a^0 = 1$

## Exponential functions IV

### General Interpretation:

- If  $a = 1 + \frac{p}{100}$ , where  $p > 0$ , and  $A > 0$ , then  $f(t)$  will increase by  $p\%$  for each unit increase in  $t$ .
- If  $a = 1 - \frac{p}{100}$ , where  $0 < p < 100$ , and  $A > 0$ , then  $f(t)$  will decrease by  $p\%$  for each unit increase in  $t$ .

# Exponential functions V

## Example

Assume that you invest €10 at an annual interest rate of 1%. Determine  $f(t)$ , the amount you have  $t$  years from now. How long does it take (approximately) for your investment to double, how long to quadruple?

# The logarithm I

The doubling time of an exponential function  $f(t) = Aa^t$  was defined as the time it takes for  $f(t)$  to become twice as large. In order to find the doubling time  $t^*$ , we must solve the equation  $a^{t^*} = 2$  for  $t^*$ :

## Logarithm function

For any positive number  $x$ ,

$$a^{\log_a x} = x$$

Thus,  $\log_a x$  is the power of  $a$  you need to get  $x$ :

$$f : \begin{cases} \mathbb{R}^+ \setminus \{0\} & \rightarrow \mathbb{R} \\ x & \mapsto \log_a x \end{cases}$$

Note: Sometimes, we write  $\log$  (or  $\ln$ ). This is the notation for  $\log_e x$ .

## The logarithm II

### Example

Find  $\ln 1$ ,  $\ln e$ ,  $\ln 1/e$ ,  $\ln 4$ , and  $\ln(-6)$ .

## The logarithm III

---

Some important rules for logarithms for  $x, y > 0$  and  $p \in \mathbb{R}$ :

L1  $\log xy =$

L2  $\log x^p =$

L3  $\ln e^x =$

L4  $e^{\ln x} =$

What is the rule for  $\ln(x) - \ln(y)$ ?

## The logarithm IV

### Example

Simplify  $\exp(\ln x^2 - 2 \ln y)$ .

# The logarithm V

## Example

How long does it take for an amount  $x$  to double at a yearly interest rate of  $i \in \{1, 2, 3\}$  per cent? Verify the “rule of 70”!

## The logarithm VI

### Example

Find the mistake in the following “proof”, showing that 2 is smaller than 1.

$$1/4 < 1/2 \Leftrightarrow$$

$$\ln(1/4) < \ln(1/2) \Leftrightarrow$$

$$\ln((1/2)^2) < \ln(1/2) \Leftrightarrow$$

$$2 \ln(1/2) < \ln(1/2) \Leftrightarrow$$

$$2 < 1$$