Test Exam

1. (4P) Let $U \sim \mathcal{U}_{(-1,1)}$ and define $X := \sqrt{1-U^2}$.
   (a) Compute the (cumulative) distribution function of X.
   (b) Derive the density function of X from (a).

2. (3P) Let $(X_n)_{n\geq 1}$ be a sequence of independent, identically distributed random variables with $X_n \sim \mathcal{U}_{(0,1)}$.
   Show
   $$\lim_{n\to\infty} \frac{\sum_{k=1}^n X_k}{\sum_{k=1}^n X_k^3} = 2$$
   almost surely.

3. (3P) Let $\varphi_X(t) := (\cos(t))^n$ be the characteristic function of some random variable X.
   (a) Show that X is discrete and identify the law of $\dfrac{X+n}{2}$ using φ_X.
   (b) Compute the expectation of X.

   Hint: $2\cos(t) = \exp(it) + \exp(-it)$

4. (6P) Let X be non-negative and $0 < \mathbb{E}[X^2] < +\infty$.
   (a) Prove that
   $$\mathbb{P}(X > 0) \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}.$$
   (b) Consider now $X \sim Poi(\lambda)$, $\lambda > 0$. Use (a) to show
   $$\exp(\lambda) \geq 1 + \lambda.$$

   Hint: Consider $\mathbb{E}[X\mathbb{1}_{\{X>0\}}]^2$ *and apply a famous inequality.*

5. (4P) Let $(Z_n)_{n\geq 1}$ be a sequence of random variables, such that for $0 < \alpha < 2$
   $$\mathbb{P}(Z_n = 0) = 1 - 2n^{-\alpha}, \qquad \mathbb{P}(Z_n = \pm n) = n^{-\alpha}.$$
   Show, $(Z_n)_{n\geq 0}$ converges in probability to 0, but does not converge to zero in L^2.